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Remarks on a Result of L.A.V. Carvalho

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The first author is Professor of Mathematics at the University of Maryland, Baltimore County. His collaboration with the second author spans a period of 10 years. The Point of Contact for this report is Malcolm Taylor, U.S. Army Research Laboratory, ATTN: AMSRL-CI-S, Aberdeen Proving Ground, MD 21005-5067.

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The report comments on two papers in the *Journal of Mathematical Analysis and Applications* and contains two new propositions that supplement results that were previously published by the authors in BRL-TR-2702, BRL-TR-2762, and ARO-Report 87-01. The note was submitted for publication in the *Journal of Mathematical Analysis and Applications* and is presently under review.

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Preface.

The note "Remarks on a Result of L. A. V. Carvalho" was submitted for publication in the *Journal of Mathematical Analysis and Applications* and is presently under review. It is published here since the new Proposition A and B supplement results that appeared in the Technical Reports BRL-TR-2702, BRL-TR-2762, and in ARO-Report 87-1.

We began to collaborate under the US Army Summer Faculty Research and Engineering Program in 1983. At that time "Chaos" was in its first bloom and we knew very little about it. We decided *ab ovo* to find out what had been proved in the case of a continuous function of one real variable and, more importantly, to analyze the proof techniques that were used under the mere assumption of continuity. Beyond this "narrow focus" no program was established. The usual long preliminary process of getting to the essential core in a new subject was considerably shortened by Targonski's Studia Mathematica Skript 6 *Topics in Iteration Theory.* The book had been acquired by the BRL-Library at the recommendation of the BRL-mathematician R. E. Shear. After we had completed the "required reading" we noticed that certain proofs by contradiction could be replaced by constructive proofs. In particular, reasoning with predecessors, especially of fixed points, became a viable proof strategy. In time and in the context of periodic loops and elementary periodic orbits the notions of $L(\infty)$ and $E(\infty)$ as infinite preorbits were formalized and found their "rightful" places in the Sarkovskii ordering, places, where previously had been only "etc. dots". These and other results were presented at Army Conferences in 1984 (Rensselaer), 1986 (Cornell), 1987 (West Point), 1988 (Colorado), 1991 (ARO-Durham) and informally discussed at other scientific meetings (Marseille-Luminy (1989), College Park (1991)).

We wish to thank Messrs. H. L. Reed, S. S. Wolff and A. B. Cooper for their support of our endeavors in the "early years" and Messrs. W. H. Mermagen and M. A. Hirschberg for their sponsorship in the "later years". We also take this opportunity to thank Messrs. Mermagen and Hirschberg for their patience during the writing of our paper "New Proof and Extension of Sarkovskii's Theorem" which went through a time-consuming monotone sequence of improvements in an attempt to achieve a "best possible" measure of lucidity.

1 June 1993

N. P. Bhatia
W. O. Egerland

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In [1] Carvalho introduced the notion of a periodic n-step orbit for a continuous function $f : R \to R$. By definition, f has a periodic n-step orbit $(x_0, x_1, \cdots, x_{n-1})$ if $x_n = x_0 < x_1 < \cdots < x_{n-1}$, where $x_{k+1} = f(x_k)$, $k = 0, 1, \cdots, n-1$. He proves that if f has a periodic n-step orbit, then f has a periodic $(n-1)$-step orbit. Invernizzi obtains the same result in a recent note [2] via Miranda's theorem of 1940.

We wish to point out : (a) Periodic n-step orbits were already introduced under the name of n-periodic loops in [3]; (b) If f has an $(n+1)$-periodic loop and $n \geq 3$, then f has two distinct n-periodic loops as shown in [3; Corollary 5.3]; (c) Carvalho's extension of Sarkovskii's order is only one of many corollaries to Theorem (SR), the principal result of [3]. More importantly, we introduced in [3] the notion of an infinite loop. By definition, f has an infinite loop if there exists $x_0 \in R$ with an infinite preorbit $(x_0, x_{-1}, \cdots, x_{-n}, \cdots)$ satisfying $x_0 < \cdots < x_{-n} < x_{-(n-1)} < \cdots < x_{-2} < x_{-1}$ or $x_0 > \cdots > x_{-n} > x_{-(n-1)} > \cdots > x_{-2} > x_{-1}$, where $f(x_{-n}) = x_{-(n-1)}$. It follows then [3; Theorem 5.4] that if f has an infinite loop, then f has n-periodic loops of all orders in at least two distinct copies for each $n \geq 3$. Furthermore, the notion of an infinite loop, as already shown in [3], is equivalent with the notion of turbulence introduced by Block and Coppel in [4], following a suggestion of Lasota and Yorke.

We conclude this note by proving two propositions. Proposition A gives a proof that an $(n+1)$-periodic loop implies the existence of two distinct n-periodic loops if $n \geq 3$. This proof is different from the previous proof

[3; Corollary 5.3]. Proposition B states a four-point inequality that ensures the existence of an infinite loop and hence two distinct n-periodic loops for each $n \geq 3$.

Proposition A. Let $f : R \to R$ be continuous. If f has an $(n+1)$-periodic loop, $n \geq 3$, then f has two distinct n-periodic loops.

Proof. For the proof we state first a lemma.
Lemma F. Let $f : R \to R$ be continuous and $L_1, L_2, \ldots, L_n$ compact intervals such that

$$f(L_i) \supset L_{i+1},\ i = 1, 2, \ldots, n-1$$

and

$$f(L_n) \supset L_1.$$

If $L_i \cap L_j$, $i \neq j$, is either empty or a singleton, then there is a point $x_0 \in L_1$ with $x_i \in L_{i+1}$, $i = 1, 2, \ldots, n-1$, and $x_0 = x_n$. Such an x_0 has period n if n is odd and period n or $\frac{n}{2}$ if n is even, but the period $\frac{n}{2}$ is possible only if $x_i \in L_{i+1} \cap L_{(\frac{n}{2}+i+1)}$, $i = 0, 1, 2, \ldots, \frac{n}{2} - 1$.
Lemma F is proved in [5]. The proof of Proposition A then follows for any $n \geq 3$ from the construction exhibited in the following Fig. 1 for the special case $n = 4$, where x_0, x_1, x_2, x_3 and x_4 are the points of a 5-periodic loop, c_0 is a fixed point, and c_{-1}, c_{-2}, and c_{-3} are predecessors of c_0.

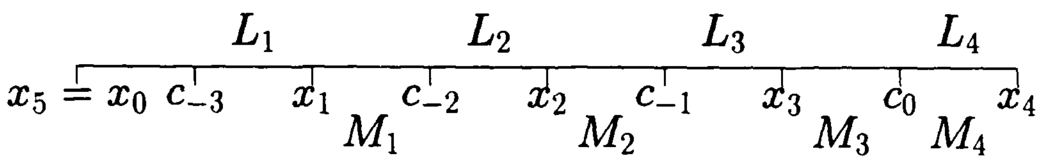

Fig. 1. Any 5-periodic loop implies two 4-periodic loops.

The intervals $L_1 = [c_{-3}, x_1]$, $L_2 = [c_{-2}, x_2]$, $L_3 = [c_{-1}, x_3]$, $L_4 = [c_0, x_4]$, and $M_1 = [x_1, c_{-2}]$, $M_2 = [x_2, c_{-1}]$, $M_3 = [x_3, c_0]$, $M_4 = [c_0, x_4]$ ensure the existence of two distinct 4-periodic loops by Lemma F.
Remark. Carvalho [1] and Invernizzi [2] prove only the existence of the 4-periodic loop defined by the L-intervals. We observe that the 4-periodic loop defined by the M-intervals does not have a 4-periodic point in the interval (x_0, x_1).

Proposition B. Let $f : R \to R$ be continuous. If

$$x_{n+1} < x_n < x_0 < x_1$$

for some $n \geq 2$, then $L(\infty)$ holds, i.e., f has an infinite loop.

Proof. The proof is based on Lemma 4.1 in [3] which states that if c_0 is a fixed point of f with predecessors c_{-1}, c_{-2} satisfying $c_0 < c_{-2} < c_{-1}$, then $L(\infty)$ holds. We shall construct a six-point inequality that contains the assumptions of Lemma 4.1. We observe first that there is a fixed point c_0 of f satisfying $x_n < c_0 < x_0$, and we may clearly choose it so that the interval $(c_0, x_0]$ contains no other fixed points of f. Hence $f(x) > x$ for each $x \in (c_0, x_0]$. We note next that there is a successor x_k of x_0, $1 \leq k \leq n-1$, such that $x_{k+1} < c_0 < x_0 < x_k$. This inequality implies the existence of another fixed point d_0 satisfying $x_0 < d_0 < x_k$, and we may assume that $f(x) < x$ in the interval $(d_0, x_k]$. Our final observation is that there must be an x_l, $0 \leq l \leq k-1$, such that $x_0 < x_l < d_0$, $x_{l+1} \geq x_k$. This completes the construction

$$x_{k+1} < c_0 < x_l < d_0 < x_k \leq x_{l+1},$$

which guarantees predecessors c_{-1} and c_{-2} in the intervals (d_0, x_k) and (x_l, d_0), respectively, with $c_0 < c_{-2} < c_{-1}$. Hence, by Lemma 4.1, $L(\infty)$ holds and the proof is complete.

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